Published by Institute of Physics Publishing for SISSA

Received: *July 13, 2007*
Accepted: *August 30, 2007*
Published: *September 10, 2007*

Two centered black holes and $N = 4$ dyon spectrum

Ashoke Sen

Harish-Chandra Research Institute
Chhatnag Road, Jhusi, Allahabad 211019, India
E-mail: sen@mri.ernet.in

Abstract: The exact spectrum of dyons in a class of N=4 supersymmetric string theories is known to change discontinuously across walls of marginal stability. We show that the change in the degeneracy across the walls of marginal stability can be accounted for precisely by the entropy of two centered small black holes which (dis)appear as we cross the walls of marginal stability.

Keywords: Black Holes in String Theory, D-branes, p-branes, Superstrings and Heterotic Strings.

JHEP09(2007)045

© SISSA 2007 http://jhep.sissa.it/archive/papers/jhep092007045/jhep092007045.pdf

Contents

1. **Introduction** **1**

2. **Prediction from exact dyon spectrum** **2**

3. **Two centered small black holes** **5**

4. **Conclusion** **8**

JHEP09(2007)045

1. Introduction

We now have a good understanding of the exact spectrum of a class of quarter BPS dyons in $\mathcal{N}=4$ supersymmetric string theories, obtained by taking an asymmetric $\mathbb{Z}_N$ orbifold of heterotic or type IIA string theory compactified on T^6 [1–13]. It is also known that as we cross various walls of marginal stability associated with the possible decay of the dyon into a pair of half BPS states, the degeneracy changes by a certain amount that is exactly computable [11]. On the other hand the asymptotic expansion of the degeneracy formula for large charges reproduces the entropy of the corresponding black hole not only to the leading order, but also to the first subleading order in an expansion in inverse power of the charges [14, 4, 7–9]. Given this correspondence between dyon spectrum and black hole entropy, a natural question to ask would be: can we understand the jump in the degeneracy across walls of marginal stability on the black hole side?

The question is somewhat tricky since these jumps in the degeneracy are exponentially small compared to the leading contribution to the entropy [11]. Nevertheless since the change is discontinuous, one might hope that there is a clear mechanism on the black hole side which produces these discontinuous changes across the walls of marginal stability and if we can identify this mechanism then we may be able to reproduce these jumps on the black hole side. In this paper we shall show that there is indeed a clear mechanism on the black hole side that describes these jumps, — this is the phenomenon of (dis)appearance of multicentered black hole solutions for a given total charge as we cross various walls of marginal stability in the space of asymptotic values of the moduli fields [15–19]. In particular the exponential of the entropy associated with these multi-centered black holes will reproduce the jump in the degeneracy computed from the exact dyon spectrum.

The role of multi-centered black holes in the context of exact dyon spectrum of $\mathcal{N}=4$ supersymmetric string theories has been discussed before in [12]. In this paper the authors considered a special class of dyonic states for which there is no single centered black hole solution but whose degeneracy is predicted to be non-zero by the exact formula, and showed how such states may be represented as 2-centered black holes. However for the charge vector used in [12] each of these two black holes had entropy of order unity, and hence their role in producing the correct contribution to the degeneracy was not manifest. In contrast we consider a dyonic state with large charges for which the change in the degeneracy across the wall of marginal stability is exponentially large (even though it is exponentially small compared to the leading contribution). The 2-centered black hole whose (dis)appearance across the wall of marginal stability is responsible for this jump is a pair of small black holes each carrying large charges and hence large entropy [20 − 28]. Thus one can calculate the entropy associated with this two centered black hole by using standard techniques and compare it with the logarithm of the jump in the degeneracy across the walls of marginal stability. The result turns out to be a perfect agreement.

2. Prediction from exact dyon spectrum

Our starting point will be heterotic or type IIA string theory compactified on $T^4 \times \widehat{S}^1 \times S^1$ modded out by a $\mathbb{Z}_N$ group. The action of the $\mathbb{Z}_N$ group involves $1/N$ unit of translation along S^1, together with an order N transformation acting on the degrees of freedom associated with T^4 and also (in the case of heterotic theory) on the internal left-moving degrees of freedom. The $\mathbb{Z}_N$ action is chosen so that it commutes with all the supersymmetries appearing from the right-moving sector of the world-sheet but (in case of type IIA string theory) projects out all the supersymmetries coming from the left-moving sector. In this theory we shall consider dyons carrying momentum $(n', \widehat{n})$, winding $(-w', -\widehat{w})$, Kaluza-Klein monopole charges $(N', \widehat{N})$ and H-monopole charges $(-W', -\widehat{W}')$ along S^1 and $\widehat{S}^1$ respectively. Such a dyon will be labelled by the electric and magnetic charge vectors

$$Q = \begin{pmatrix} \widehat{n} \\ n' \\ \widehat{w} \\ w' \end{pmatrix}, \qquad P = \begin{pmatrix} \widehat{W} \\ W' \\ \widehat{N} \\ N' \end{pmatrix}. \tag{2.1}$$

The precise sign convention used for defining these charges can be found in [13]. We shall denote by M the symmetric SO(2, 2) matrix that encodes information about the moduli labelling the torus $\widehat{S}^1 \times S^1$ and by $a + iS$ the axion-dilaton modulus. If we denote by the subscript $_\infty$ the asymptotic values of the various moduli, then the quarter BPS dyon of charge (Q, P) can decay into a pair of half BPS states of charges $(Q, 0)$ and $(0, P)$ on the wall of marginal stability [11]:

$$a_\infty + \frac{(P^T(M_\infty + L)Q)}{\left[(Q^T(M_\infty + L)Q)(P^T(M_\infty + L)P) - (P^T(M_\infty + L)Q)^2\right]^{1/2}} S_\infty = 0\,, \tag{2.2}$$

where

$$L = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \tag{2.3}$$

is the SO(2, 2) invariant matrix. There are other walls of marginal stability associated with the decay into other pairs of half-BPS states [11] but we shall carry out our analysis in the vicinity of the wall (2.2). Other cases may be analyzed in the same way.

JHEP09(2007)045

We shall consider diagonal M_∞ of the form:

$$M_\infty = \begin{pmatrix} \widehat{R}^{-2} & & & \\ & R^{-2} & & \\ & & \widehat{R}^2 & \\ & & & R^2 \end{pmatrix} . \tag{2.4}$$

In this case $\widehat{R}$ and R can be interpreted as the radii of $\widehat{S}^1$ and S^1 respectively, measured in units of $\sqrt{\alpha'}$. We shall also focus on a special class of dyons for which[1]

$$Q = \begin{pmatrix} 0 \\ -n/N \\ 0 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} Q_1 - 1 \\ -J \\ Q_5 \\ 0 \end{pmatrix}, , \qquad n, J, Q_1, Q_5 \in \mathbb{Z}, \quad n, Q_1 \geq 0, \quad Q_5 > 0 \,, \tag{2.5}$$

since for these states the exact degeneracy — more precisely an index that counts the number of bosonic minus the number of fermionic supermultiplets[2] — can be computed by using a dual type IIB description [7–9]. In this case (2.2) takes the form:

$$a_\infty = a_c, \quad a_c \equiv -\frac{J\,\widehat{R}}{R\{Q_1 - 1 + \widehat{R}^2 Q_5\}}\, S_\infty \,. \tag{2.6}$$

The weak coupling region of the dual type IIB string theory corresponds to the large R region in the current description [11]. In this region the degeneracy formula takes the form [7–9] (see [11] for a review of the results):

$$d(Q,P) = \begin{cases} d_>(Q,P) & \text{for } a_\infty > a_c \\ d_<(Q,P) & \text{for } a_\infty < a_c \end{cases} , \tag{2.7}$$

where

$$\begin{aligned} d_>(\vec{Q},\vec{P}) &= \frac{1}{N}\int_{\mathcal{C}_>} d\widetilde{\rho}\, d\widetilde{\sigma}\, d\widetilde{v}\, e^{-\pi i(N\widetilde{\rho}Q^2 + \widetilde{\sigma}P^2/N + 2\widetilde{v}Q\cdot P)}\, \frac{1}{\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})} \,, \\ d_<(\vec{Q},\vec{P}) &= \frac{1}{N}\int_{\mathcal{C}_<} d\widetilde{\rho}\, d\widetilde{\sigma}\, d\widetilde{v}\, e^{-\pi i(N\widetilde{\rho}Q^2 + \widetilde{\sigma}P^2/N + 2\widetilde{v}Q\cdot P)}\, \frac{1}{\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})} \,. \end{aligned} \tag{2.8}$$

Here

$$Q^2 = Q^T L Q = 2n/N, \qquad P^2 = P^T L P = 2Q_5(Q_1 - 1), \qquad Q\cdot P = Q^T L P = J \,, \tag{2.9}$$

[1]By following the procedure given in [13] we could switch on non-zero values of the first and third components of Q, but in order to keep the various formulæ simple we shall continue to work with the charge vector given in (2.5).

[2]The degeneracy $d(\vec{Q},\vec{P})$ given in (2.7) actually refers to the number of bosonic minus fermionic supermultiplets multiplied by a factor of $(-1)^{Q\cdot P+1}$. The $(-1)^{Q\cdot P+1}$ factor was not included in the analysis of [7–9, 11]. The $(-1)^{Q\cdot P}$ factor appeared in [29] and reflects the change in statistics in going from a five to four dimensional viewpoint in the presence of a Kaluza-Klein monopole. The additional $-$ sign appears in the study of the bound state of a D1-D5 system to a Kaluza-Klein monopole [30, 31]. These will be discussed in detail in a forthcoming review [32].

JHEP09(2007)045

$\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})$ is a known function of three complex variables $(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})$ and $C_>$ and $C_<$ are a pair of three real dimensional subspaces of the three complex dimensional space labelled by $(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) \equiv (\widetilde{\rho}_1 + i\widetilde{\rho}_2, \widetilde{\sigma}_1 + i\widetilde{\sigma}_2, \widetilde{v}_1 + i\widetilde{v}_2)$. They are defined as

$$\begin{aligned} C_> : &\ \widetilde{\rho}_2 = M_1, \quad \widetilde{\sigma}_2 = M_2, \quad \widetilde{v}_2 = -M_3, \\ &\ 0 \le \widetilde{\rho}_1 \le 1, \quad 0 \le \widetilde{\sigma}_1 \le N, \quad 0 \le \widetilde{v}_1 \le 1\,, \\ C_< : &\ \widetilde{\rho}_2 = M_1, \quad \widetilde{\sigma}_2 = M_2, \quad \widetilde{v}_2 = M_3, \\ &\ 0 \le \widetilde{\rho}_1 \le 1, \quad 0 \le \widetilde{\sigma}_1 \le N, \quad 0 \le \widetilde{v}_1 \le 1\,, \end{aligned} \tag{2.10}$$

M_1, M_2 and M_3 being large but fixed positive numbers with $M_3 \ll M_1, M_2$. For $\widetilde{v} \simeq 0$, $\widetilde{\Phi}$ takes the form:

$$\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) = -4\pi^2\, \widetilde{v}^2\, f(N\widetilde{\rho}) g(\widetilde{\sigma}/N) + \mathcal{O}(\widetilde{v}^4)\,, \tag{2.11}$$

where $(f(\tau))^{-1}$ and $(g(\tau))^{-1}$ have the interpretation of the generating function for the degeneracies of purely electric half-BPS states and purely magnetic half-BPS states respectively. For example for the $\mathbb{Z}_N$ orbifold of the heterotic string theory on $T^4 \times \widehat{S}^1 \times S^1$ with prime values of N we have [4]

$$f(\tau) = (\eta(\tau/N))^{k+2}\eta(\tau)^{k+2}, \qquad g(\tau) = (\eta(\tau))^{k+2}\eta(N\tau)^{k+2}\,, \qquad k \equiv \frac{24}{N+1} - 2\,. \tag{2.12}$$

For $N = 1$, i.e. for heterotic string theory on $T^4 \times \widehat{S}^1 \times S^1$, this gives us back the standard result $\eta(\tau)^{24}$ for both $f(\tau)$ and $g(\tau)$.

The jump in the degeneracy as we move from $a_\infty < a_c$ to $a_\infty > a_c$ is determined by an integral over the difference between the contours $C_>$ and $C_<$. The contribution to this integral comes from the pole of the integrand at $\widetilde{v} = 0$ [11]. Substituting (2.11) into (2.8) and evaluating the residue at the pole at $\widetilde{v} = 0$ we get

$$d_>(Q,P) - d_<(Q,P) = -Q \cdot P\, d_{\rm el}(Q)\, d_{\rm mag}(P)\,, \tag{2.13}$$

where

$$d_{\rm el}(Q) = \int_0^1 d\widetilde{\rho}\, e^{-i\pi N\widetilde{\rho}Q^2}\, (f(N\widetilde{\rho}))^{-1}\,, \qquad d_{\rm mag}(P) = \frac{1}{N}\int_0^N d\widetilde{\sigma}\, e^{-i\pi\widetilde{\sigma}P^2/N}\, (g(\widetilde{\sigma}/N))^{-1}\,, \tag{2.14}$$

are the degeneracies of purely electric and purely magnetic half-BPS states carrying charges Q and P respectively. Thus $\ln d_{\rm el}(Q)$ and $\ln d_{\rm mag}(P)$ are the entropies of small black holes of electric charge Q and magnetic charge P respectively. Since $\ln|Q \cdot P|$ is subleading compared to these entropies for large Q^2 and P^2 i.e. for

$$n, Q_1, Q_5 \gg 1\,, \tag{2.15}$$

we see that $\ln|d_>(Q,P) - d_<(Q,P)|$ can be identified as the sum of the entropies of a small electric black hole of charge Q and a small magnetic black hole of charge P. In carrying out the analysis on the black hole side we shall choose charge vectors satisfying (2.15).

Taking into account the sign of the right hand side of (2.13), and assuming that this phenomenon has a description in the dual black hole picture, we can draw the following

JHEP09(2007)045

conclusion:[3] *For $J(= Q\cdot P) > 0$, as we cross the wall of marginal stability* (2.6) *from $a_\infty > a_c$ to $a_\infty < a_c$, a new configuration should appear whose entropy is equal to the sum of the entropies of a small electric black hole of charge Q and a small magnetic black hole of charge P. On the other hand for $J(= Q\cdot P) < 0$, as we cross the wall of marginal stability* (2.6) *from $a_\infty < a_c$ to $a_\infty > a_c$, a new configuration should appear whose entropy is equal to the sum of the entropies of a small electric black hole of charge Q and a small magnetic black hole of charge P.*

In section 3 we shall verify this explicitly by identifying the new configuration as a two centered black hole solution with an electric center of charge vector Q and a magnetic center of charge vector P.

3. Two centered small black holes

For describing the two centered black hole we shall use the $\mathcal{N} = 2$ supersymmetric description of the same system described above. In the supergravity approximation the relevant part of the theory is described by the prepotential (see [34] for a review):

$$F = -\frac{X^1 X^2 X^3}{X^0}\,, \tag{3.1}$$

where X^I's are scalar fields. These are related to the scalar fields $a + iS$ and M via the relations

$$a + iS = \frac{X^1}{X^0}, \quad T = -i\frac{X^2}{X^0}, \quad U = -i\frac{X^3}{X^0}\,, \tag{3.2}$$

iT and iU being the Kahler and complex structure modulus of the torus $\widehat{S}^1 \times S^1$. They contain the same information as the matrix M. In particular for the asymptotic M given in (2.4), we have

$$T_\infty = R\widehat{R}, \quad U_\infty = \widehat{R}/R\,. \tag{3.3}$$

The theory contains four gauge fields, and we shall denote the electric and magnetic charges associated with these gauge fields by q_0, q_1, q_2, q_3 and p^0, p^1, p^2, p^3 respectively. These charges can be related to the charge vectors Q and P introduced earlier via the relation:

$$Q = \begin{pmatrix} q_0 \\ q_3 \\ -p^1 \\ q_2 \end{pmatrix}, \qquad P = \begin{pmatrix} q_1 \\ p^2 \\ p^0 \\ p^3 \end{pmatrix}. \tag{3.4}$$

Thus for the configuration (2.5) we have

$$(q_0, q_1, q_2, q_3) = (0, Q_1 - 1, -1, -n/N), \qquad (p^0, p^1, p^2, p^3) = (Q_5, 0, -J, 0)\,. \tag{3.5}$$

[3] There are two points to note here. First when a new configuration with same charge appears in the black hole system, its degeneracy (or more precisely the index), i.e. exponential of the entropy, will add to the degeneracy of the other configurations of the same charge. Second, we shall be implicitly assuming that the new system that appears gives a positive contribution to $d(\vec{Q}, \vec{P})$. Otherwise the condition on $Q\cdot P$ stated in the proposal will be reversed. With the sign convention for $d(\vec{Q}, \vec{P})$ described in footnote 2 this assumption is consistent with the wall crossing formula of [33, 18].

JHEP09(2007)045

The theory has an underlying gauge invariance that allows for a scaling of all the X^I's by a complex function. We shall fix this gauge using the gauge condition:

$$i(\bar{X}^I F_I - X^I \bar{F}_I) = 1\,, \qquad F_I \equiv \partial F/\partial X^I\,, \tag{3.6}$$

which amounts to setting $\alpha' = 8$. This fixes the normalization but not the overall phase of the X^I's. While studying a black hole solution carrying a given set of charges, it will be convenient to fix the overall phase of the X^I's such that

$$Arg(q_I X^I - p^I F_I) = \pi \quad \text{at } \vec{r} = \infty\,. \tag{3.7}$$

In this gauge one can construct a general multi-centered black hole solution with charges $(q^{(s)}, p^{(s)})$ located at $\vec{r}_s$. The locations $\vec{r}_s$ are constrained by the equations [15–17]

$$h_I p^{(s)I} - h^I q_I^{(s)} + \sum_{t\neq s} \frac{p^{(s)I} q_I^{(t)} - q_I^{(s)} p^{(t)I}}{|\vec{r}_s - \vec{r}_t|} = 0 \tag{3.8}$$

where h^I and h_I are constants defined through the equations

$$X^I_\infty - \bar{X}^I_\infty = i h^I\,, \qquad F_{I\infty} - \bar{F}_{I\infty} = i h_I\,. \tag{3.9}$$

If we define α and β via the relations

$$X^0_\infty = \alpha + i\beta\,, \tag{3.10}$$

then using (3.1)-(3.3) and (3.9) we get

$$\begin{gathered} h^0 = 2\beta, \quad h^1 = 2(\beta a_\infty + \alpha S_\infty), \quad h^2 = 2\widehat{R} R\alpha, \quad h^3 = 2\widehat{R}\alpha/R, \\ h_0 = -2\widehat{R}^2(\alpha S_\infty + \beta a_\infty), \quad h_1 = 2\beta \widehat{R}^2, \quad h_2 = 2\widehat{R}(\beta S_\infty - \alpha a_\infty)/R, \\ h_3 = 2\widehat{R}R(\beta S_\infty - \alpha a_\infty)\,. \end{gathered} \tag{3.11}$$

The gauge condition (3.6) gives

$$\alpha^2 + \beta^2 = (8\widehat{R}^2 S_\infty)^{-1}\,. \tag{3.12}$$

To proceed further we need to focus on a specific multi-centered solution. Since our goal is to identify a configuration whose entropy is the sum of the entropies of a purely electric small black hole of charge Q and a purely magnetic small black hole of charge P, the natural object to focus on is a two centered solution with electric charge Q at one center and a magnetic charge P at the other center. This will automatically have the desired entropy.[4] Using (2.5), (3.4) we see that the charges at the two centers are given by:

$$q^{(1)} = (0,0,-1,-n/N), \;\; p^{(1)} = (0,0,0,0), \;\; q^{(2)} = (0, Q_1 - 1, 0, 0), \;\; p^{(2)} = (Q_5, 0, -J, 0)\,. \tag{3.13}$$

[4] In the supergravity approximation the solution is singular at each center, but once higher derivative corrections are taken into account each center is transformed into the near horizon geometry of a non-singular extremal black hole with finite entropy equal to the statistical entropy of the corresponding microstates. This has been demonstrated explicitly for the $\mathbb{Z}_N$ orbifolds of heterotic string theory on $T^4 \times \widehat{S}^1 \times S^1$ [20–28]. In this case the modifications of the solution due to higher derivative corrections can be found using the method developed in [35]. This approach fails for type II string compactification, most likely due to the absence of an AdS_3 factor in the near horizon geometry of the small black hole. However it is expected that once the effect of full set of higher derivative terms are taken into account the entropy of a small black hole in type II string theory will also reproduce the statistical entropy of the corresponding microstates.

JHEP09(2007)045

eqs.(3.8) for $s = 1$ and 2 now gives:

$$h^2 + \frac{n}{N} h^3 = \frac{J}{L}, \tag{3.14}$$

$$h_0 Q_5 - h_2 J - h^1 (Q_1 - 1) + \frac{J}{L} = 0\,, \tag{3.15}$$

where $L = |\vec{r}_1 - \vec{r}_2|$ is the separation between the two centers. Using (3.11) and (3.14) we get

$$\alpha = \frac{J}{2L} \frac{1}{R\widehat{R} + \frac{n}{N}\frac{\widehat{R}}{R}}\,. \tag{3.16}$$

Using (3.11) and (3.16) we may now express (3.15) as

$$\beta\left(a_\infty(Q_1 - 1 + \widehat{R}^2 Q_5) + \frac{\widehat{R} J S_\infty}{R}\right) + \alpha\left((Q_1 - 1 + \widehat{R}^2 Q_5) S_\infty - \widehat{R} R - \frac{n}{N}\frac{\widehat{R}}{R} - \frac{\widehat{R} J a_\infty}{R}\right) = 0\,. \tag{3.17}$$

Substituting (3.16), (3.17) into (3.12) we can determine L. The ambiguity in determining the sign of L can be fixed using (3.7).

We are interested in determining under what conditions the two centered black hole solution described above exists. For this we note that a sensible solution should have positive value of L. Typically as we change the values of the asymptotic moduli keeping the charges fixed, the value of L changes. On some subspace of codimension 1 the value of L becomes infinite and beyond that the solution gives negative values of L which means that the solution does not exist. To determine this codimension 1 subspace we simply need to determine the conditions on the asymptotic moduli for which $L = \infty$. From (3.16) we see that in this case $\alpha = 0$. Since eq.(3.12) now requires β to be non-zero, we see from (3.17) that

$$a_\infty (Q_1 - 1 + \widehat{R}^2 Q_5) + \widehat{R} J S_\infty / R = 0\,. \tag{3.18}$$

This is identical to the condition (2.6) for marginal stability [15]. Thus we conclude that as a_∞ passes through a_c, the two centered black hole solution carrying an entropy equal to the sum of the entropies of a small electric black hole of charge Q and a small magnetic black hole of charge P, (dis)appears from the spectrum. This is precisely what was predicted at the end of section 2 by analyzing the exact formula for the degeneracy of dyons.

In order to complete the verification of the predictions made at the end of section 2 we need to determine on which side of the $a_\infty = a_c$ line the two centered solution exists. For this we use eq.(3.7). For the solution under consideration this gives, using (3.17),

$$\alpha\left(a_\infty(Q_1 - 1 + \widehat{R}^2 Q_5) + \frac{\widehat{R} J S_\infty}{R}\right)\left\{1 + \frac{\left((Q_1 - 1 + \widehat{R}^2 Q_5) S_\infty - \widehat{R} R - \frac{n}{N}\frac{\widehat{R}}{R} - \frac{\widehat{R} J a_\infty}{R}\right)^2}{\left(a_\infty(Q_1 - 1 + \widehat{R}^2 Q_5) + \frac{\widehat{R} J S_\infty}{R}\right)^2}\right\} < 0\,. \tag{3.19}$$

First consider the case $J > 0$. Since L must be positive for the two centered solution to exist, we see from (3.16) that $\alpha > 0$. In this case the term on the left hand side of (3.19) is negative for $a_\infty < a_c$ and positive for $a_\infty > a_c$. Thus the inequality is satisfied only

JHEP09(2007)045

for $a_\infty < a_c$, leading to the conclusion that the two centered black hole exists only for $a_\infty < a_c$. A similar analysis shows that for $J < 0$, the two centered black hole exists only for $a_\infty > a_c$. This is exactly what has been predicted at the end of section 2 from the analysis of the exact dyon spectrum of the theory.

4. Conclusion

The main conclusion that can be drawn from the analysis of this paper is that the exact formula for the degeneracy of dyons in $\mathcal{N} = 4$ supersymmetric string theories encodes information not only about the single centered black holes, but also about the multi-centered black holes whose total charge adds up to that of the dyon whose degeneracy is under consideration. Since in the present example the contribution to the degeneracy from the two centered black holes is exponentially small compared to that from the single centered black hole, our results indicate that the correspondence between the microscopic degeneracy of states and black hole entropy extends beyond the leading asymptotic expansion, — not only for terms which are suppressed by inverse powers of charges but also for terms which are exponentially suppressed.

Note added: the relation between the two centered black holes and the jump in the degeneracy in $\mathcal{N} = 4$ dyon spectrum has also been discussed in [36] which appered a few days after this paper was first submitted to the arXiv.

References

[1] R. Dijkgraaf, E.P. Verlinde and H.L. Verlinde, *Counting dyons in $N = 4$ string theory*, *Nucl. Phys.* **B 484** (1997) 543 [hep-th/9607026].

[2] D. Shih, A. Strominger and X. Yin, *Recounting dyons in $N = 4$ string theory*, *JHEP* **10** (2006) 087 [hep-th/0505094].

[3] D. Gaiotto, *Re-recounting dyons in $N = 4$ string theory*, hep-th/0506249.

[4] D.P. Jatkar and A. Sen, *Dyon spectrum in CHL models*, *JHEP* **04** (2006) 018 [hep-th/0510147].

[5] J.R. David, D.P. Jatkar and A. Sen, *Product representation of dyon partition function in CHL models*, *JHEP* **06** (2006) 064 [hep-th/0602254].

[6] A. Dabholkar and S. Nampuri, *Spectrum of dyons and black holes in CHL orbifolds using borcherds lift*, hep-th/0603066.

[7] J.R. David and A. Sen, *Chl dyons and statistical entropy function from D1-D5 system*, *JHEP* **11** (2006) 072 [hep-th/0605210].

[8] J.R. David, D.P. Jatkar and A. Sen, *Dyon spectrum in $N = 4$ supersymmetric type-II string theories*, *JHEP* **11** (2006) 073 [hep-th/0607155].

[9] J.R. David, D.P. Jatkar and A. Sen, *Dyon spectrum in generic $N = 4$ supersymmetric $Z(N)$ orbifolds*, *JHEP* **01** (2007) 016 [hep-th/0609109].

[10] A. Dabholkar and D. Gaiotto, *Spectrum of CHL dyons from genus-two partition function*, hep-th/0612011.

JHEP09(2007)045

[11] A. Sen, *Walls of marginal stability and dyon spectrum in $N = 4$ supersymmetric string theories*, *JHEP* **05** (2007) 039 [hep-th/0702141].

[12] A. Dabholkar, D. Gaiotto and S. Nampuri, *Comments on the spectrum of CHL dyons*, hep-th/0702150.

[13] N. Banerjee, D.P. Jatkar and A. Sen, *Adding charges to $N = 4$ dyons*, arXiv:0705.1433.

[14] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Asymptotic degeneracy of dyonic $N = 4$ string states and black hole entropy*, *JHEP* **12** (2004) 075 [hep-th/0412287].

[15] F. Denef, *Supergravity flows and D-brane stability*, *JHEP* **08** (2000) 050 [hep-th/0005049].

[16] F. Denef, B.R. Greene and M. Raugas, *Split attractor flows and the spectrum of BPS D-branes on the quintic*, *JHEP* **05** (2001) 012 [hep-th/0101135].

[17] B. Bates and F. Denef, *Exact solutions for supersymmetric stationary black hole composites*, hep-th/0304094.

[18] F. Denef and G.W. Moore, *Split states, entropy enigmas, holes and halos*, hep-th/0702146.

[19] F. Denef and G.W. Moore, *How many black holes fit on the head of a pin?*, arXiv:0705.2564.

[20] A. Dabholkar, *Exact counting of black hole microstates*, *Phys. Rev. Lett.* **94** (2005) 241301 [hep-th/0409148].

[21] A. Dabholkar, R. Kallosh and A. Maloney, *A stringy cloak for a classical singularity*, *JHEP* **12** (2004) 059 [hep-th/0410076].

[22] A. Sen, *How does a fundamental string stretch its horizon?*, *JHEP* **05** (2005) 059 [hep-th/0411255].

[23] V. Hubeny, A. Maloney and M. Rangamani, *String-corrected black holes*, *JHEP* **05** (2005) 035 [hep-th/0411272].

[24] D. Bak, S. Kim and S.-J. Rey, *Exactly soluble BPS black holes in higher curvature $N = 2$ supergravity*, hep-th/0501014.

[25] A. Sen, *Black holes, elementary strings and holomorphic anomaly*, *JHEP* **07** (2005) 063 [hep-th/0502126].

[26] A. Dabholkar, F. Denef, G.W. Moore and B. Pioline, *Exact and asymptotic degeneracies of small black holes*, *JHEP* **08** (2005) 021 [hep-th/0502157].

[27] A. Dabholkar, F. Denef, G.W. Moore and B. Pioline, *Precision counting of small black holes*, *JHEP* **10** (2005) 096 [hep-th/0507014].

[28] P. Kraus and F. Larsen, *Microscopic black hole entropy in theories with higher derivatives*, *JHEP* **09** (2005) 034 [hep-th/0506176].

[29] D. Shih and X. Yin, *Exact black hole degeneracies and the topological string*, *JHEP* **04** (2006) 034 [hep-th/0508174].

[30] C.N. Pope, *Axial vector anomalies and the index theorem in charged Schwarzschild and Taub-Nut spaces*, *Nucl. Phys.* **B 141** (1978) 432.

[31] J.P. Gauntlett, N. Kim, J. Park and P. Yi, *Monopole dynamics and BPS dyons in $N = 2$ super- Yang-Mills theories*, *Phys. Rev.* **D 61** (2000) 125012 [hep-th/9912082].

JHEP09(2007)045

[32] A. Sen, *Black hole entropy function, attractors and precision state counting*, to appear.

[33] F. Denef, *Quantum quivers and Hall/hole halos*, *JHEP* **10** (2002) 023 [hep-th/0206072].

[34] T. Mohaupt, *Black hole entropy, special geometry and strings*, *Fortschr. Phys.* **49** (2001) 3 [hep-th/0007195].

[35] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Stationary BPS solutions in $N = 2$ supergravity with R^2 interactions*, *JHEP* **12** (2000) 019 [hep-th/0009234].

[36] M.C.N. Cheng and E. Verlinde, *Dying dyons don't count*, arXiv:0706.2363.

JHEP09(2007)045